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# On Minimal Non*PN*-Groups

N. S. NARASIMHA SASTRY

*Department of Mathematics and Statistics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*

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## 1. INTRODUCTION

A well-known theorem of Wielandt states that a finite group $G$ is nilpotent if and only if every maximal subgroup of $G$ is normal in $G$. The structure of a nonnilpotent group, each of whose proper subgroups is nilpotent, has been analyzed by Schmidt and Rédei [5, Satz 5.1 and Satz 5.2, pp. 280–281]. In [1], Buckley investigated the structure of a $PN$-group (i.e., a finite group in which every minimal subgroup is normal), and proved (i) that a $PN$-group of odd order is supersolvable, and (ii) that certain factor groups of a $PN$-group of odd prime power order are also $PN$-groups. Earlier, Gaschütz and Itô [5, Satz 5.7, p. 436] had proved that the commutator subgroup of a finite $PN$-group is $p$-nilpotent for each odd prime $p$. This paper is a sequel to [9] and our object here is to prove the following statement.

THEOREM. *If $G$ is a finite non$PN$-group, each of whose proper subgroups is a $PN$-group, then one of the following statements is true*:

(a) *$G$ is the dihedral group of order* 8.

(b) $G = \langle a, x \colon a^{p^n} = 1, x^p = 1,$ *and* $x^{-1}ax = a^{1+p^{n-1}} \rangle$, *$p$ an odd prime.*

(c) $G = \langle a, b, x \colon a^{p^n} = b^p = x^p = 1$ *for some prime $p$*, $[x, a] = b$, *and* $[a, b] = [b, x] = 1\rangle$ (*i.e.*, $G = M\langle x\rangle$, *where* $M = \langle a, b\rangle \simeq Z_{p^n} \times Z_p$, $[a, x] = b$ *and* $[x, b] = 1$).

(d) $G = PQ$, *where* $P \lhd G$, $P = \langle x\rangle^G \in \mathrm{Syl}_p(G)$ *is elementary abelian, and* $Q \in \mathrm{Syl}_q(G)$ *is nonnormal and cyclic.*

(e) $G = P\langle x\rangle$, *where $P$ is an ultraspecial* 2-*group of order* $2^{3s}$ *and* $|x|$ *is a prime dividing* $2^s + 1$.

We recall that a $p$-group $P$ is called ultraspecial if $P' = \Phi(P) = Z(P) = \Omega_1(P)$. While the examples of type (d) are easily constructed, Hawkes [4, Theorem 5.3,



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p. 221] has constructed examples of type (e) corresponding to each prime dividing $2^s + 1$. Our notation and terminology is standard and is mostly adopted from [3]. In addition, $M \lessdot G$ denotes that $M$ is a maximal subgroup of $G$.

## 2. Preliminaries

In this section, we collect some of the results used later. The trivial remark (2.1) is used often and without reference. (2.2) is well known and the proof is omitted.

(2.1) Let $G$ be a minimal non$PN$-group, and let $\langle t \rangle$ be a minimal subgroup of $G$.

(i) If $t$ is contained in more than one maximal subgroup of $G$, then $\langle t \rangle \lhd G$.

(ii) If $t$ is contained in a unique maximal subgroup $M(t)$ of $G$, and if $H$ is a proper subgroup of $G$ containing $t$, then $H \subseteq M(t)$.

(2.2) (i) $PSL(3, 3)$ contains a maximal subgroup $M$ of order $2^4 3^3$.

(ii) A Sylow 3-subgroup of $PSL(3, 3)$ is a nonabelian group of order $3^3$ and exponent 3.

(iii) If $S$ is a Sylow 3-subgroup of $PSL(3, 3)$ contained in $M$ and $Z(S) = \langle x \rangle$, then $\langle x \rangle \ntriangleleft M$.

(2.3) Let $P$ be a $p$-group, and let $P$ be abelian if $p = 2$. If $\alpha$ is an automorphism of $P$ fixing every element of $P$ of order $p$, then $| \alpha |$ is a power of $p$.

*Proof.* This is [6, Hilfsatz 1.8, p. 51].

(2.4) (Thompson [8]; see also [5, Bermerkung 7.5, p. 190]) Any finite nonabelian simple group, all of whose proper subgroups are solvable, is isomorphic to one of the following simple groups:

(a) $PSL(3, 3)$;
(b) $PSL(2, p)$, where $p$ is a prime with $p > 3$ and $p^2 - 1 \not\equiv 0(5)$;
(c) $PSL(2, 2^q)$, where $q$ is a prime;
(d) $PSL(2, 3^q)$, where $q$ is an odd prime;
(e) The Suzuki group $Sz(2^q)$, where $q$ is an odd prime.

(2.5) If $X$ is any one of the simple groups mentioned in (2.4) other than $PSL(3, 3)$, then $X$ is a Zassenhaus group of degree $n + 1$, where $n = r$ or $r^2$ according as $X = PSL(2, r)$ or $X = Sz(r)$; and the stabilizer $N$ of a point is a maximal subgroup of $X$. Further, $N$ is a Frobenius group with kernel $K$ of order $n$ and a cyclic complement $H$. If $X = PSL(2, r)$, then $| H | = (r - 1)/d$, where $d = (r - 1, 2)$; and if $X = Sz(r)$, then $| H | = r - 1$. Also, $N' = K$

and $H$ contains a Sylow $p$-subgroup of $X$ for any odd prime divisor $p$ of $|H|$ with $(p, r) = 1$.

*Proof.* See [3, Theorem 8.2, p. 41; 7, pp. 224–227].

## 3. Proof of the Theorem

### A. Solvability of G

Suppose that $G$ is nonsolvable. Then $G = G'$, because every proper subgroup of $G$ is solvable [5, Satz 5.7, p. 436].

(i) $\langle 1 \rangle \neq \Phi(G) = \mathrm{Core}_G(M)$ for each $M \lessdot G$.

*Proof.* Since $N_G(M) = M$, $M \cap M^g \neq \langle 1 \rangle$ for some $g \in G \backslash M$ [3, Theorems 7.7 and 7.6(i), pp. 38–39] and $\mathrm{Core}_G(M) \neq \langle 1 \rangle$. Clearly, $\Phi(G) \subseteq \mathrm{Core}_G(M)$. If $\mathrm{Core}_G(M) \nsubseteq M_1$ for some $M_1 \lessdot G$, then $G = M_1 \,\mathrm{Core}_G(M)$ would be solvable. Hence, $\Phi(G) = \mathrm{Core}_G(M)$.

(ii) If $t \in G$ is of prime order, then either $t \in Z(G)$ or $t$ is contained in a unique maximal subgroup $M(t)$ of $G$.

*Proof.* $G = G'$ and (2.1) imply (ii).

(iii) If $t$ and $M(t)$ are as in (ii), then

(a) $t \in P$ and $P \in \mathrm{Syl}_p(G)$ imply that $P \subseteq M(t)$,

(b) $\langle t, t^g \rangle = G$ for each $g \in G \backslash M(t)$, and

(c) $|t| \neq 2$.

*Proof.* The uniqueness of $M(t)$ implies (a) and (b). If $|t| = 2$, then $G = \langle t, t^g \rangle$ would be a dihedral group and hence solvable. Thus, (c) follows.

(iv) Let $p$ be an odd prime, and let $M \lessdot G$.

(a) If $|M|_p < |G|_p$, then $M$ is $p$-nilpotent.

If $P \in \mathrm{Syl}_p(G)$ and $P \subseteq M$, then

(b) $M = M(t)$ for some element $t$ of $P$ of order $p$, $t \in Z(P)$ and $\langle t \rangle \lhd M$,

and

(c) $P \subseteq M'$.

*Proof.* If $|M|_p < |G|_p$, then, by (ii) and (iii(a)), every element of $M$ of order $p$ lies in the center $Z(G)$ of $G$. Now (a) follows from [5, Satz 5.5, p. 435].

If $P \subseteq M$, then $M = M(t)$ for some element $t$ of order $p$; otherwise, $G$ would be $p$-nilpotent by (ii) and [5, Satz 5.5, p. 435]. Since $\langle t \rangle$ is weakly closed in $P$

with respect to $G$, we deduce using Grün's theorem [2, Theorem 4.7, p. 10] that $P = P \cap G = P \cap G' = P \cap (N_G(\langle t \rangle))' = P \cap M'$. Hence, (c) follows.

(v) By (i), $G/\Phi(G)$ is a chief factor of $G$. Since every proper subgroup of $G$ is solvable but $G$ is not, it follows that $G/\Phi(G)$ is isomorphic to one of the simple groups listed in (2.4).

(vi) Now we verify that none of the simple groups in (2.4) can be isomorphis to $G/\Phi(G)$.

Suppose that $G/\Phi(G) \simeq PSL(3, 3)$, and let $M_1 \lessdot G$ correspond to the maximal subgroup $M$ mentioned in (2.2) above. Since $M$ contains a Sylow 3-subgroup $S$ of $PSL(3, 3)$, $M_1$ also contains a Sylow 3-subgroup $S_1$ of $G$. Hence, by (iv(b)), $M_1 = M_1(t)$ for some element $t \in S_1$ of order 3. Since $t \in Z(S_1)$ and since $Z(S) = \langle x \rangle$ has order 3, $\langle t \rangle$ maps onto $\langle x \rangle$ in $PSL(3, 3)$. But the normality of $\langle t \rangle$ in $M_1$ implies the normality of $\langle x \rangle$ in $M$, a contradiction to (2.2(iii)). Therefore, $G/\Phi(G) \not\simeq PSL(3, 3)$.

Suppose that $G/\Phi(G)$ is isomorphic to some simple group $X \neq PSL(3, 3)$ in (2.4), and let us adopt the notation of (2.5). The last assertion in (2.5) and (iv(c)) imply that no odd prime divides $r - 1$. This immediately rules out the cases (2.4(c)) and (2.4(e)).

If $X$ is as in (2.4(b)), then $p = 1 + 2^n$ for some natural number $n$. Since $PSL(2, 5) \simeq A_5$ contains $A_4$ as a maximal subgroup, (iv(c)) implies that $G/\Phi(G) \not\simeq PSL(2, 5)$. Therefore, $n \geqslant 4$. Since $p^2 - 1 \equiv 0(16)$, $X$ contains $S_4$ [5, Hauptsatz 8.27, p. 213]. By checking the list of subgroups of $X$, we conclude that $S_4 \lessdot X$. Clearly, 3 divides $| X | = (p - 1) p(p + 1)/2$, and, since $S_4$ is not 3-nilpotent, (iv(a)) implies that $3^2$ does not divide $| X |$. Hence, $S_4$ contains a Sylow 3-subgroup of $X$. If $M \lessdot G$ corresponds to the maximal subgroup $S_4$ of $X$, then, by (iv(b)), $M = M(t)$ for some element $t$ of order 3. Since $\langle t \rangle \lhd M$ and $t \notin \Phi(G)$, the image of $\langle t \rangle$ in $X$ is the normal Sylow 3-subgroup of $S_4$. This impossibility proves that $X$ cannot be as in (2.4(b)).

If $X$ is as in (2.4(d)), then $3^q - 1 = 2^t$ for some $t \geqslant 4$. Since $3^{2q} - 1 \equiv 0(16)$, $S_4 \lessdot X$ [5, Hauptsatz 8.27, p. 213]. But then (iv(a)) implies that $S_4$ is 3-nilpotent. Therefore, $G/\Phi(G) \not\simeq PSL(2, 3^q)$. This completes the proof of the solvability of $G$.

Let $\langle x \rangle$ be a minimal nonnormal subgroup of $G$, and let $\{P_1, ..., P_r\}$ be a Sylow system of $G$ with, say, $x \in P_1$. If $r > 2$, then $P_1 P_i < G$ and $N_G(\langle x \rangle) \supseteq \langle P_1, ..., P_r \rangle = G$, a contradiction. Therefore, $| G |$ is divisible by at most two primes. Let $p$ and $q$ denote distinct primes.

## *B. The Case when G is a p-Group*

In this case, $G = M\langle x \rangle$, where $M \lessdot G$ and $\langle x \rangle$ is a nonnormal subgroup of order $p$. Also, $N = N_G(\langle x \rangle) = C_G(\langle x \rangle) \lessdot G$. Let $a \in M \backslash M \cap N$. Since

$a \notin C_G(\langle x \rangle)$ and $M \cap N \lessdot M$, it follows that $G = \langle a, x \rangle$, $a^p \in M \cap N \subseteq C_G(\langle x \rangle)$, and hence, that $a^p \in Z(G)$. Since $G$ is generated by two elements and since $G/M \cap N \simeq Z_p \times Z_p$, we have that $M \cap N = \Phi(G)$. Since $[[a, x], x] = 1$, $[a, x]^p = [a, x^p] = 1$; and, since $[a, x] \neq 1$, $|[a, x]| = p$. Therefore, $a \in C_G(\langle [a, x] \rangle)$, $[a, x] \in Z(G)$, and $G' = \langle [a, x]^g : g \in G \rangle = \langle [a, x] \rangle$. Hence $G^p = \langle a^p \rangle$ and $\Phi(G) = G'G^p = \langle [a, x], a^p \rangle \subseteq Z(G)$. Since $G$ is nonabelian and $[G : \Phi(G)] = p^2$, $Z(G) = \Phi(G)$ and every maximal subgroup of $G$ is abelian. Now statements (a), (b), and (c) in the theorem correspond to the cases when $[a, x] \in \langle a \rangle$ and $p = 2$, $[a, x] \in \langle a \rangle$ and $p \neq 2$, and $[a, x] \notin \langle a \rangle$, respectively.

*C. The Case when* $|G| = p^a q^b$, *where* $a \neq 0$ *and* $b \neq 0$

Since $G$ is solvable, $G$ has a normal maximal subgroup $H$ of index, say, $q$. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ so that $H = P(Q \cap H)$. Since $H$ is a $PN$-group, $\Omega_1(H) = \Omega_1(P)\,\Omega_1(Q \cap H)$ char $H \lhd G$ and is elementary abelian. Therefore, $\Omega_1(P)$ and $\Omega_1(Q \cap H)$ are normal in $G$.

*The case when* $\Omega_1(P) = P$. If each minimal subgroup of $P$ is normal in $G$, then some minimal subgroup $\langle x \rangle$ of $Q$ is nonnormal in $G$. Since $x \in Q$, $P\langle x \rangle = G$ and $|P| = p$.

If $P$ contains a nonnormal minimal subgroup $\langle x \rangle$ of $G$, then $\langle x \rangle^G = \Omega_1(P)$ is elementary abelian and $G/\langle x \rangle^G$ is a cyclic $q$-group.

Both of these cases correspond to statement (d) of the theorem.

*The case when* $\Omega_1(P) < P$. In this case, $\Omega_1(H)Q = \Omega_1(P)Q < G$ and $\Omega_1(H)$ is contained in more than one maximal subgroup of $G$. Since $G$ is a non$PN$-group and $P \subseteq H$, $G = H\langle x \rangle$ for some nonnormal minimal subgroup $\langle x \rangle$ of $G$ of order $q$.

Suppose that $\langle x, x^g \rangle \subsetneq G$ for each $g \in G$. Then $[x, x^g] = 1$ and $\langle x \rangle^G \subsetneq Q$. If $\langle x \rangle^G \subsetneq Q$, then $P\langle x \rangle^G \subsetneq G$ and $N_G(\langle x \rangle) \supseteq \langle P, Q \rangle = G$, a contradiction. Therefore, $\langle x \rangle^G = Q \lhd G$. Since $x$ and, hence, $Q$ are contained in a unique maximal subgroup of $G$, $G/Q \simeq P$ is cyclic. Now $Q \cap H$ char $H \lhd G$, $[P, x] \neq \langle 1 \rangle$, and $[x, \Omega_1(P)] = 1$. If bars denote the images in $\bar{G} = G/Q \cap H$, then $\bar{x} \neq \bar{1}$, $\bar{P} = \bar{H} \lhd \bar{G}$, $[\bar{x}, \bar{P}] \neq \bar{1}$, and $[\bar{x}, \Omega_1(\bar{P})] = 1$. Hence, by (2.3), $\bar{x}$ is a $p$-element, a contradiction. Therefore, $\langle x, x^g \rangle = G$ for some $g \in G$. If $q = 2$, then $G$ is a dihedral group. So, we can assume that $q \neq 2$.

Now $x \notin F(G)$, since $G = \langle x, x^g \rangle$ is nonnilpotent; and $C_G(F(G)) \subseteq F(G)$ and $\langle x \rangle \ntriangleleft G$ imply that $G = F(G)\langle x \rangle$ and $F(G) = P \in \mathrm{Syl}_p(G)$. Since $[P, x] \neq 1$ and $[\Omega_1(P), x] = 1$, it follows from (2.3) that $P$ is a nonabelian 2-group. Since $[x, N] = 1$ for each $\langle x \rangle$-invariant subgroup $N$ of $P$, and since $x$ is contained in a unique maximal subgroup of $G$, it follows that $\langle x \rangle$ acts irreducibly on $P/\Phi(P)$ and $[P, x] = P$. Now we use Theorems 1.3 and 2.2 of [4] to conclude that statement (e) of the theorem holds in this case.

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## REFERENCES

1. J. BUCKLEY, Finite groups whose minimal subgroups are normal, *Math. Zeit.* **116** (1970), 15–17.
2. G. GLAUBERMAN, Global and local properties of finite groups, *in* "Finite Simple Groups" (M. B. Powell and G. Higman, Eds.), Academic Press, New York/London, 1971.
3. D. GORENSTEIN, "Finite Groups," Harper & Row, New York, 1968.
4. T. HAWKES, On the automorphism group of a 2-group, *Proc. London Math. Soc.* **26** (1973), 207–225.
5. B. HUPPERT, "Endliche Gruppen I," Springer-Verlag, New York, 1967.
6. B. HUPPERT, Subnormale Untergruppen und $p$-Sylow-gruppen, *Acta Sci. Math. (Szeged)* **22** (1961), 46–61.
7. N. ITÔ, Frobenius and Zassenhaus groups, lecture notes, Department of Mathematics, University of Illinois at Chicago Circle, 1968–1969.
8. J. G. THOMPSON, Nonsolvable finite groups all of whose local subgroups are solvable, I, *Bull. Amer. Math. Soc.* **74** (1968), 383–437.
9. N. S. N. SASTRY AND W. E. DESKINS, Influence of normality conditions of almost minimal subgroups of a finite group, *J. Algebra* **52** (1978), 364–377.